Topic: Reference angle theorem

Question: What is the reference angle for the given angle?

$\frac{37}{5}\pi$ radians

Answer choices:

A $\frac{1}{5}\pi$

B $\frac{2}{5}\pi$

C $\frac{3}{5}\pi$

D $\frac{4}{5}\pi$

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Solution: B

Let $\theta = (37/5)\pi$. We'll first find an angle α in the interval $(0,2\pi)$ such that the terminal side of α coincides with the terminal side of θ.

Well,

$$\frac{37}{5}\pi = \frac{(7+30)}{5}\pi = \frac{7}{5}\pi + \frac{30}{5}\pi = \frac{7}{5}\pi + 6\pi = \frac{7}{5}\pi + 2\pi(3)$$

Let $\alpha = (7/5)\pi$. Then θ and α differ by an integer multiple of 2π, so the terminal side of θ coincides with the terminal side of α. Moreover,

$$\pi < \frac{7}{5}\pi = \left(1\frac{2}{5}\right)\pi = 1.4\pi < 1.5\pi = \frac{3}{2}\pi$$

Therefore, α is in the third quadrant, so it's below the negative x-axis (which we can represent by an angle of measure π), hence the reference angle for α is

$$\frac{7}{5}\pi - \pi = \frac{2}{5}\pi$$

Since the terminal side of θ coincides with the terminal side of α, we know that the reference angle for θ is equal to the reference angle for α. Hence the reference angle for θ is $(2/5)\pi$.

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Topic: Reference angle theorem

Question: Choose the best approximation.

If θ is an angle in the second quadrant such that $\cos\theta = -0.713$, which of the following most closely approximates the value of $\sin(\theta + 5\pi)$?

Answer choices:

A 0.508

B -0.713

C -0.701

D 0.693

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Solution: C

First, we'll compute $\sin\theta$:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + (-0.713)^2 = 1$$

$$\sin^2\theta = 1 - (-0.713)^2$$

Now $(-0.713)^2 \approx 0.508$, so

$$\sin^2\theta \approx 1 - 0.508$$

$$\sin^2\theta \approx 0.492$$

We were given that θ is in the second quadrant, so $\sin\theta$ is positive and

$$\sin\theta \approx \sqrt{0.492}$$

$$\sin\theta \approx 0.701$$

Since θ is in the second quadrant, we can "think" of θ as being an angle in the interval $(\pi/2,\pi)$, that is,

$$\frac{1}{2}\pi < \theta < \pi$$

Thus θ is above the negative x-axis (which we can represent by an angle of measure π), so its reference angle is $\pi - \theta$.

Now note that

$$\theta + 5\pi = \theta + (4\pi + \pi) = (\theta + 4\pi) + \pi$$

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Since θ is in the second quadrant (and we're "thinking" of θ as having positive measure), an angle of measure $\theta + 5\pi$ is equivalent to (starting on the positive x-axis and) undergoing a rotation of θ in the positive direction (which brings us to θ), followed by two complete turns around the origin in the positive direction (to get from θ to $\theta + 4\pi$) and an additional rotation of π in the positive direction (to get from $\theta + 4\pi$ to $\theta + 5\pi$).

Denote $\theta + 4\pi$ by θ', and note that the terminal side of θ' coincides with the terminal side of θ (because θ and $\theta + 4\pi$ differ by an integer multiple of 2π). Therefore, the terminal side of $\theta' + \pi$ coincides with the terminal side of $\theta + \pi$. Note that

$$\theta' + \pi = (\theta + 4\pi) + \pi = \theta + 5\pi$$

That is, the terminal side of $\theta + 5\pi$ coincides with the terminal side of $\theta + \pi$, so the reference angle for $\theta + 5\pi$ is equal to the reference angle for $\theta + \pi$.

Since θ is in the second quadrant, $\theta + \pi$ is in the fourth quadrant, so $\theta + \pi$ is below the negative x-axis (which we can represent by an angle of measure 2π). Now recall that we're "thinking" of θ as being in the interval $(\pi/2, \pi)$:

$$\frac{1}{2}\pi < \theta < \pi$$

Adding π throughout, we obtain

$$\frac{1}{2}\pi + \pi < \theta + \pi < \pi + \pi$$

$$\frac{3}{2}\pi < \theta + \pi < 2\pi$$

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Therefore, the reference angle for $\theta + \pi$ (and hence the reference angle for $\theta + 5\pi$) is

$$2\pi - (\theta + \pi) = 2\pi - \theta - \pi = \pi - \theta$$

This is equal to the reference angle for θ, which we found earlier. Thus by the reference angle theorem, together with the fact that θ is in the second quadrant and $\theta + 5\pi$ is in the fourth quadrant, we find that

$$\sin(\theta + 5\pi) = -\sin(\theta) = -0.701$$

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Topic: Reference angle theorem

Question: If θ is an angle such that $\sin\theta = 0.439$, what are the two possible values of $\cos(\theta + 540°)$?

Answer choices:

A $0.193, \ -0.193$

B $0.807, \ -0.807$

C $0.327, \ -0.327$

D $0.898, \ -0.898$

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Solution: D

We're given the value of $\sin\theta$, so we'll first find the value of $\cos\theta$:

$$\sin^2\theta + \cos^2\theta = 1$$

Substituting the value of $\sin\theta$, we have

$$(0.439)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - (0.439)^2$$

Now $(0.439)^2 \approx 0.193$, so

$$\cos^2\theta \approx 1 - 0.193$$

$$\cos^2\theta \approx 0.807$$

Well,

$$\sqrt{0.807} \approx 0.898$$

so either

$$\cos\theta \approx 0.898$$

or

$$\cos\theta \approx -0.898$$

Since $\sin\theta$ is positive, θ is in either the first quadrant or the second quadrant. We'll first consider the case where θ is in the first quadrant, which implies that $\cos\theta$ is positive, hence

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$$\cos\theta \approx 0.898$$

Since θ is in the first quadrant, we can "think" of θ as being in the interval $(0°,90°)$, so θ is above the positive x-axis (which we can represent by an angle of measure 0). Therefore, the reference angle for θ is $\theta - 0 = \theta$.

Now note that

$$\theta + 540° = \theta + (360° + 180°) = (\theta + 360°) + 180°$$

Since θ is in the first quadrant (and we're "thinking" of θ as having positive measure), an angle of measure $\theta + 540°$ is equivalent to (starting on the positive x-axis and) undergoing a rotation of θ in the positive direction (which brings us to θ), followed by one complete turn around the origin in the positive direction (to get from θ to $\theta + 360°$) and an additional rotation of $180°$ in the positive direction (to get from $\theta + 360°$ to $\theta + 540°$).

Denote $\theta + 360°$ by θ', and note that the terminal side of θ' coincides with the terminal side of θ (because θ and $\theta + 360°$ differ by an integer multiple of $360°$). Therefore, the terminal side of $\theta' + 180°$ coincides with the terminal side of $\theta + 180°$. Note that

$$\theta' + 180° = (\theta + 360°) + 180° = \theta + 540°$$

That is, the terminal side of $\theta + 540°$ coincides with the terminal side of $\theta + 180°$, so the reference angle for $\theta + 540°$ is equal to the reference angle for $\theta + 180°$.

Since θ is in the first quadrant, we know that $\theta + 180°$ is in the third quadrant, so $\theta + 180°$ is below the negative x-axis (which we can represent

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by an angle of measure $180°$). Now recall that we're "thinking" of θ as being in the interval $(0°,90°)$:

$$0° < \theta < 90°$$

Adding $180°$ throughout, we obtain

$$0° + 180° < \theta + 180° < 90° + 180°$$

$$180° < \theta + 180° < 270°$$

Therefore, the reference angle for $\theta + 180°$ (and hence the reference angle for $\theta + 540°$) is

$$(\theta + 180°) - 180° = \theta + 180° - 180° = \theta$$

This is equal to the reference angle for θ, which we found earlier. Thus by the reference angle theorem, together with the fact that θ is in the first quadrant and $\theta + 540°$ is in the third quadrant, we find that

$$\cos(\theta + 540°) = -\cos(\theta) \approx -0.898$$

Now let's consider the case where θ is in the second quadrant, which implies that $\cos\theta$ is negative, hence

$$\cos\theta \approx -0.898$$

Therefore, the terminal side of θ intersects the unit circle at the point

$$(x, y) = (\cos\theta, \sin\theta) \approx (-0.898,0.439)$$

Now let α denote the first-quadrant angle θ that we discussed earlier, and note that α intersects the unit circle at the point

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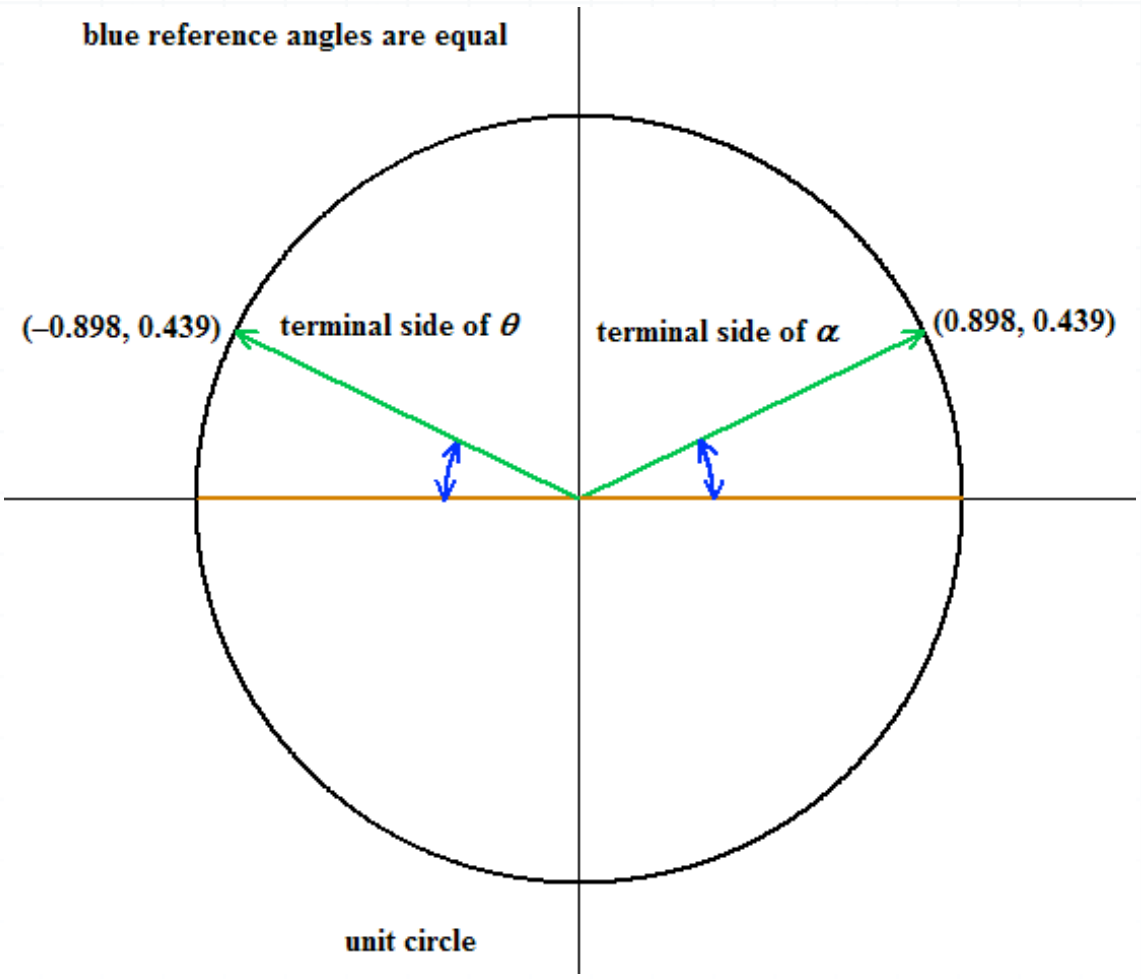
$$(x, y) = (\cos\alpha, \sin\alpha) \approx (0.898, 0.439)$$

Let's draw θ and α on the same set of coordinate axes.



By symmetry, we can see that the reference angle for θ is equal to the reference angle for α (which was α itself, since α is in the first quadrant and we "thought" of α as being in the interval $(0°, 90°)$).

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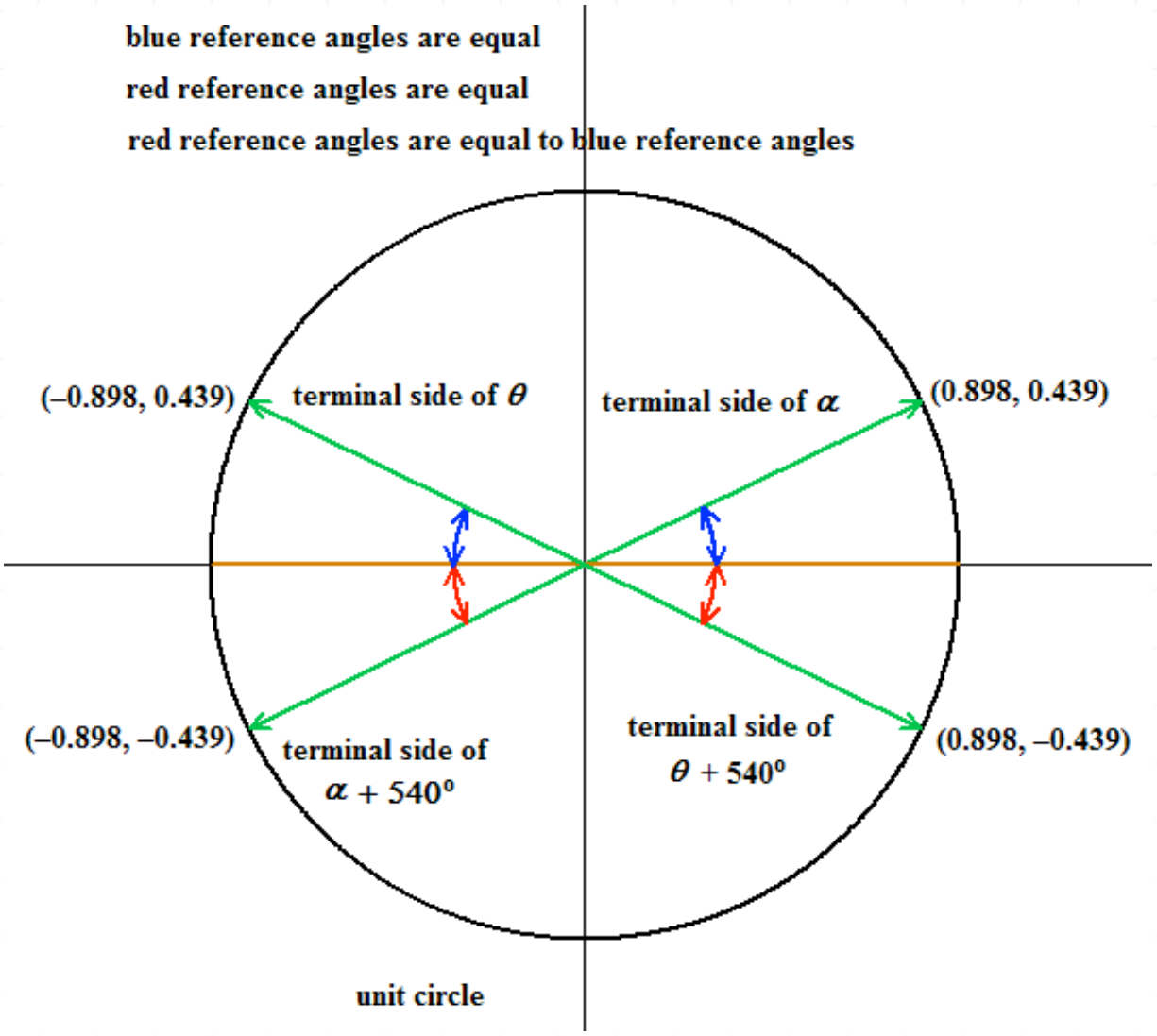
blue reference angles are equal
red reference angles are equal
red reference angles are equal to blue reference angles
(–0.898, 0.439)
terminal side of θ
terminal side of α
(0.898, 0.439)
(–0.898, –0.439)
terminal side of $\alpha + 540°$
terminal side of $\theta + 540°$
(0.898, –0.439)
unit circle

If we were to go through the corresponding process for θ that we did for α, we would find that the terminal side of $\theta + 540°$ is in the fourth quadrant. We would also find that the reference angle for $\theta + 540°$ is equal to the reference angle for $\theta + 180°$, and that the reference angle for $\theta + 180°$ is equal to the reference angle for θ (which we found to be α). Since θ is in the second quadrant and $\theta + 540°$ is in the fourth quadrant, the reference angle theorem tells us that

$$\cos(\theta + 540°) = -\cos\theta = 0.898$$

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